Algebraic Algorithm for Positive-Definite Quadratic Programming Models (QPM) with Linear and Quadratic Constraints

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Abstract

The paper develops a quadrex algorithm for quadratic programming problems (n=2) under linear and quadratic constraints. The quadrex algorithm centers on the behavior of the quadratic function at the origin and performs a series of translations and orthogonal rotations to obtain the extremum of the objective function. The method works for n > 2 provided that the eigenvalues of the quadratic form part of the objective function is strictly positive or that Q is strictly positive-definite. The quadrex algorithm is the quadratic counterpart of the simplex algorithm for linear programming models (LP).

***Keywords**: simplex, quadratic, quadratic programming, quadrex, NP-hard AMS-2020134*

Introduction

Linear programming (LP) models are extensively use in many scientific and business applications. Their popularity stem from the ease with which solutions are derived via the simplex algorithm (Taha, 1997). The simplex algorithm proceeds by examining each of the corner points of the convex polyhedron constituting the set of constraints through algebraic manipulation of the simplex tableau. The mathematical simplicity of the simplex algorithm often explains why many of the inherently non-linear optimization models are "linearized". (Johnson, 2000)

For non-linear programs (NLP), no simple algebraic algorithm exists currently that could take the place of the simplex methods in LP. Even in the case of a quadratic programming problem (QP), the user needs some working knowledge of the Kuhn-Tucker conditions, Lagrange multipliers and Calculus in order to find the optimal solution, if it exists.

The general QP problem can be stated as:

$$(1) \ldots Min: z = X^T Q X + A^T X^T + K$$

Subject To:

$$X^T \mu_i x + B_i^T X^T \leq K_i \quad , i = 1, 2, \ldots, p$$

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where X is an $nx1$ vector of decision variables, Q is an $n\ x\ n$ symmetric matrix, A is an $nx1$ vector of constants and $K_i \in R$. If the matrix $M_i = 0$, then the ith constraints is a pure linear constraint. The classical approach to solve this problem is to form the Lagrangian function:

$$(2)\ ...L = z + \sum_{i=1}^{P} \lambda_i\ (X^T \mu_i x + B_i^T X^T - K_i)$$

and then set the derivatives $\frac{\partial L}{\partial x_j} \cdot \frac{\partial L}{\partial \lambda_i}, \begin{matrix} i = 1, 2, \dots, p \\ j = 1, 2, \dots, n \end{matrix}$ equal to zero.

The QP problem with linear constraints is of particular interest. The model is given by:

Minimize (with respect to **x**)

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x}.$$

Subject to one or more constraints of the form:

$A\mathbf{x} \leq \mathbf{b}$ (inequality constraint)

$E\mathbf{x} = \mathbf{d}$ (equality constraint)

where $\mathbf{x}^T$ indicates the vector transpose of $\mathbf{x}$. The notation $A\mathbf{x} \leq \mathbf{b}$ means that every entry of the vector $A\mathbf{x}$ is less than or equal to the corresponding entry of the vector $\mathbf{b}$.

If the matrix Q is positive semidefinite matrix, then f is a convex function: In this case the quadratic program has a global minimizer if there exists some feasible vector $\mathbf{x}$ (satisfying the constraints) and if f is bounded below on the feasible region. If the matrix Q is positive definite and the problem has a feasible solution, then the global minimizer is unique. If Q is zero, then the problem becomes a linear program.

Quadratic programming is particularly simple when there are only equality constraints; specifically, the problem is linear. By using Lagrange multipliers and seeking the extremum of the Lagrangian, it may be readily shown that the solution to the equality constrained problem is given by the linear system:

$$\begin{bmatrix} Q & E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

where λ is a set of Lagrange multipliers which come out of the solution alongside $\mathbf{x}$.

The easiest means of approaching this system is direct solution (for example, LU factorization), which for small problems is very practical. For large problems, the system poses some unusual difficulties, most notably that problem is never positive definite (even if Q is), making it potentially very difficult to find a good numeric approach, and there are many approaches to choose from dependent on the problem . For positive definite *Q*, the ellipsoid method solves the problem in polynomial time. If, on the other hand, Q is indefinite, then the problem is NP-hard. In fact, even if Q has only one negative eigenvalue, the problem is NP-hard (Pardalos and Vavasi, 1991).

Other methods for solving a QP is through the sequential quadratic programming method. Sequential quadratic programming (SQP) is an iterative method for nonlinear optimization. SQP methods are used on problems for which the objective function and the constraints are twice continuously differentiable. SQP methods solve a sequence of optimization subproblems, each of which optimizes a quadratic model of the objective subject to a linearization of the constraints. If the problem is unconstrained, then the method reduces to Newton's method for finding a point where the gradient of the objective vanishes. If the problem has only equality constraints, then the method is equivalent to applying Newton's method to the first-order optimality conditions, or Karush–Kuhn–Tucker conditions, of the problem.

Consider a nonlinear programming problem of the form:

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & b(x) \geq 0 \\ & c(x) = 0. \end{array}$$

The Lagrangian for this problem is

$$\mathcal{L}(x, \lambda, \sigma) = f(x) - \lambda^T b(x) - \sigma^T c(x),$$

where λ and σ are Lagrange multipliers.

At an iterate x_k, a basic sequential quadratic programming algorithm defines an appropriate search direction d_k as a solution to the quadratic programming subproblem ((Frederic, Charles et al., 2006)

$$\begin{array}{ll} \min\limits_{d} & \mathcal{L}(x_k, \lambda_k, \sigma_k) + \nabla \mathcal{L}(x_k, \lambda_k, \sigma_k)^T d + \frac{1}{2} d^T \nabla^2_{xx} \mathcal{L}(x_k, \lambda_k, \sigma_k) d \\ \text{s.t.} & b(x_k) + \nabla b(x_k)^T d \geq 0 \\ & c(x_k) + \nabla c(x_k)^T d = 0. \end{array}$$

Of course, there are other methods for solving (1) but these methods are equally inaccessible to non-mathematicians. Optimization by vector space methods, for instance, rely on embedding the NLP in an appropriate Hilbert space. The solution is the found by an application of the Projection Theorem. (D. Luenberger, 1987). It is, however, ironic that majority of the users of NLP’s are non-mathematicians who apply this programming model to business, economics, and social science problems.

This paper examines the QPM from an elementary algebra and analytic geometry perspective by developing an algorithm called **quadrex**.

The Quadrex Algorithm

We restrict our attention to the case where n = 2. The general quadratic function in two variables:

$$(3)\ldots f(x,y) = ax^2 + by^2 + cxy + dx + ey + f,\ \ c, b \neq 0$$

can be written as:

$$(4) \qquad z = f(x,y) = X^T QX + A^T X^T + K$$

where: $Q = \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}, A^T = (d,e), X = \begin{pmatrix} x \\ y \end{pmatrix}.$

We note in passing that if $z = m$, a constant, then (3) represents a conic section.

If $c = d = e = 0$, then we obtain:

$$f(x,y) = ax^2 + by^2$$

If a and b are both positive, then the origin (0, 0) is a minimum while if a and b are both negative, then (0, 0) is a maximum.

Lemma 1. Let $z = ax^2 + by^2$, then:

(i) if a, b > 0 , then (0, 0) is a minimum;
(ii) if a, b < 0 , then (0, 0) is a maximum.

Proof:

Construct a neighbourhood around the origin $N_\varepsilon(0,0) = \{(x,y) | x^2 + y^2 < \varepsilon^2\}$. Let $(X_0, Y_0) \in N_\varepsilon(0,0)$. Suppose that a, b > 0. Without loss of generality, let $a = \min\{a, b\}$. Then:

$$0 < aX_0^2 + aY_0^2 < a\varepsilon^2 < aX_0^2 + bY_0^2$$

It follows that $f(X_0, Y_0) > 0 = f(0,0)$ and the origin is therefore a minimum. The proof in the case of a maximum is similar.■

The effect of adding a linear term to Equation (5) is to shift the origin. Thus,

$$(6) \qquad z = ax^2 + by^2 + dx + ey + K$$

Shifts the origin from (0, 0) to $\left(\frac{-d}{2a}, \frac{-e}{2b}\right)$. By Lemma 1, this new origin is a minimum or a maximum depending on the signs of a and b.

Lemma 2. Let $= ax^2 + by^2 + dx + ey + K$, then

(i) if a, b > 0, the point $\left(\frac{-d}{2a}, \frac{-e}{2b}\right)$ is a minimum and $Z_{min} = K - \left(\frac{d^2}{4a} - \frac{\varepsilon^2}{4b}\right)$;

(ii) if a, b < 0, the point $\left(\frac{d}{2a}, \frac{e}{2b}\right)$ is a maximum and $Z_{max} = K + \left(\frac{d^2}{4a} + \frac{\varepsilon^2}{4b}\right)$.

Proof:

Apply Lemma 1 to the new origin. ■

The case when a and b have opposite signs is a bit more tricky. Consider the behaviour of the function:

(7) ... $z = ax^2 + by^2 \quad , a, b > 0$

at the origin. Equation (7) factors as:

(8) ...$z = (\sqrt{a}\, x + \sqrt{b}\, y)(\sqrt{a}\, x - \sqrt{b}\, y)$,

And the origin becomes a saddle point where the lines $y = \frac{\sqrt{a}}{\sqrt{b}} x$ and $y = \frac{-\sqrt{a}}{\sqrt{b}} x$ intersect.

General Unconstrained QP

The general second-degree quadratic function can be represented by Equation (4). We attempt to reduce (4) to the form (6) by rotation of axes. Since Q is symmetric, it follows that Q can be represented as:

(9) ... $Q = RDR^T$ or $D = R^T QR$

where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is a diagonal matrix whose elements are the eigenvalues of Q and R is an orthogonal matrix whose columns are the eigenvectors corresponding to the eigenvalues. Let:

(10) $X^* = RX \ or\ X = R^T X^*$, then (6) becomes:

(11) $Z = X^{*T} DX^* + (RA)^T X^* + K$
$Z = \lambda_1 x^{*2} + \lambda_2 y^{*2} + m_1 x^* + m_2 y^* + K$

which is now in the same form as equation (6).

Theorem 1: Let $Z = X^T Q\, X + A^T X^T + K$ as previously specified. Then:
(i) if Q is positive-definite, a, b > 0, then Z has a minimum obtained by Lemma 2.
(ii) if Q is positive-definite, a, b < 0, then Z has a maximum obtained by Lemma 2.

Constrained Optimization

Consider:

Minimize: $z = ax^2 + by^2 \ \ , a, b > 0$
Subject To:
$cx^2 + dy^2 < K,\ K > 0$

The global minimum (0, 0) is the origin and is a feasible solution since it satisfies the constraint. We will therefore be interested in the case when the constraint *does not include* the origin:

(12) ... Minimize: $z = ax^2 + by^2 \ \ , a, b > 0$
Subject To:
$cx^2 + dy^2 \geq K,\ K > 0$

Here, the origin (0, 0) is not a feasible solution.

Consider the arc traced by $cx^2 + dy^2 = K$. We claim that the optimal solution which is feasible lies along this arc. This arc is an ellipse. If $c > d$, the major axis of the ellipse is along the vertical axis while if $c < d$, the major axis is along the horizontal axis.

Suppose $c > d$, then the major axis has coordinates $\left(\sqrt{\frac{K}{C}}, 0\right)$ and $\left(-\sqrt{\frac{K}{C}}, 0\right)$. The minor axis has coordinates $\left(0, \sqrt{\frac{K}{d}},\right)$ and $\left(0, -\sqrt{\frac{K}{d}}\right)$. The minimum occurs along the arc at a point closest to the global (infeasible) minimum (0, 0), namely, $\left(0, \sqrt{\frac{K}{d}},\right)$ and $\left(0, -\sqrt{\frac{K}{d}}\right)$.

Theorem 2. The optimal solution to:

Minimize: $z = ax^2 + by^2 \ \ , a, b > 0$
Subject To:
$cx^2 + dy^2 \geq K,\ K > 0$
occurs at the endpoints of the minor axis of the ellipse $cx^2 + dy^2 = K$.

Proof:

Take any other point (X_0, Y_0) satisfying $c{X_0}^2 + d{Y_0}^2 = K$ where $c \ < \ d$. The value of Zat , $\left(0, \sqrt{\frac{K}{d}},\right)$ is:

$$Z^* = \frac{b\ k}{d}$$

The value of Z at (X_0, Y_0) is:

$$Z_0 = a{x_0}^2 + d{y_0}^2 > by_0 \geq b \cdot \frac{k}{d} = Z^*$$

Hence, $Z^* < Z_0$. ■

The extension of Theorem 2 to several quadratic constraints is obvious. We consider the intersections of the elliptical constraints:

$$c_i\, x^2 + d_i\, y^2 \geq k_i \quad , k_i > 0 \quad , i = 1, \dots p$$

and choose the intersection closest to the origin.

If the objective function has a linear component:

$$Z = ax^2 + by^2 + dx + ey + f$$

Then by Lemma 2, the global minimum occurs at $\left(\frac{-d}{2a}, \frac{-e}{2b}\right)$. We examine the constraints:

$$c_i\, x^2 + d_i\, y^2 \geq k_i \quad , k_i > 0$$

and determine whether the global minimum is feasible or not. Otherwise, we take the intersection point of the constraints closest to the global minimum. The search for the closest point of intersection is facilitated by the fact that it is the one which matches the signs of the components of the global minimum.

Numerical Illustrations

We provide several numerical illustrations of the quadrex algorithm.

Illustration 1. Obtain the extrema of:

1. $Z = 2x^2 + 3y^2 + 5$
2. $Z = -4x^2 - 2y^2 + 7$
3. $Z = 2x^2 + 3y^2 + 2x - 3y + 5$

Solution

1. Since a = 2, b = 3 are both positive, the minimum is at (0, 0) and the minimum Z is $Z_{min} = 5$.

2. Since a = -4, b = -2 are both negative, the maximum is at (0, 0) $Z_{max} = 7$.

3. Rewrite: $Z = 2\left(x^2 + x + \frac{1}{4}\right) + 3\left(y^2 - y + \frac{1}{4}\right) + 5 + \frac{5}{4}$

$$Z = 2\left(x + \frac{1}{2}\right)^2 + 3\left(y - \frac{1}{2}\right)^2 + \frac{25}{4}$$

The minimum occurs at $\left(-\frac{1}{2},\frac{1}{2}\right)$ and $Z_{min} = \frac{25}{4}$.

Illustration 2.

$$Min: Z = 2x^2 + 3y^2 + 2x - 3y + 5$$

Subject To:

$$2x^2 + 3y^2 \geq 2.$$

Solution:

Since the global minimum $\left(-\frac{1}{2},\frac{1}{2}\right)$ is not feasible, we locate the minor axis of the ellipse $2x^2 + 3y^2 = 2$. The endpoints are $\left(0,\sqrt{\frac{2}{3}}\right)$ *and* $\left(0,-\sqrt{\frac{2}{3}}\right)$. Of the two, $\left(0,\sqrt{\frac{2}{3}}\right)$ is closest to $\left(-\frac{1}{2},\frac{1}{2}\right)$. Since $Z = 7 - \sqrt{6}$ at $\left(0,\sqrt{\frac{2}{3}}\right)$ and $Z = 7 + \sqrt{6}$ at $\left(0,-\sqrt{\frac{2}{3}}\right)$ the optimal solution is $\left(0,\sqrt{\frac{2}{3}}\right)$.

Illustration 3.

$$Min: Z = 2x^2 + 3y^2 + 2x - 3y + 5$$

Subject To:

$$2x^2 + 3y^2 \geq 2.$$
$$2x^2 + y^2 \geq 2.$$

Solution: The intersections of the constraints are:

$$y^2 = 0 \Rightarrow y = 0$$
$$x^2 = 1 \Rightarrow x = \pm 1$$

The point closest to the global minimum is (-1, 0).

Here:

$$Z_{min} = 2(-1)^2 + 2(-1) + 5 = 5$$

while Z at (1, 0) is:

$$Z = 2(1)^2 + 2(1) + 5 = 9.$$

Illustration 4.

$$Min: Z = 2x^2 + 3y^2 + 2x - 3y + 5$$

Subject To:

$$2x^2 + y^2 \geq 8.$$
$$x^2 + 2y^2 \geq 8.$$

Solution: The intersections are at:

$$\left(\frac{\sqrt{8}}{\sqrt{3}},\frac{\sqrt{8}}{\sqrt{3}}\right),\left(\frac{\sqrt{8}}{\sqrt{3}},\frac{-\sqrt{8}}{\sqrt{3}}\right)$$
$$\left(\frac{-\sqrt{8}}{\sqrt{3}},\frac{\sqrt{8}}{\sqrt{3}}\right),\left(\frac{-\sqrt{8}}{\sqrt{3}},\frac{-\sqrt{8}}{-\sqrt{3}}\right)$$

The intersection point closest to $\left(-\frac{1}{2},\frac{1}{2}\right)$ is $\left(-\sqrt{8/3},\sqrt{8/3}\right)$. Here,

$$Z_{min} = 2\left(-\sqrt{8/3}\right)^2 + 3\left(\sqrt{8/3}\right)^2 - 2\left(-\sqrt{8/3}\right) - 3\sqrt{8/3} + 5$$
$$Z_{min} = \frac{16}{3} + \frac{24}{3} - 2\sqrt{8/3} - 3\sqrt{8/3} + 5$$

$$Z_{min} = \frac{55 - 10\sqrt{6}}{3}$$

Illustration 5.

Obtain the minimum of $Z = 2x^2 + 3y^2 + z^2 + 4x - 6y - 2z + 6$.

Solution:

We can rewrite the function as:

$$Z = 2(x + 1)^2 + 3(y-1)^2 + (z-1)^2 + 12$$

Since the coefficients of the squared variables are all positive, it follows that (-1,1,1) is a minimum and then $Z_{min} = 12$.

Conclusion

The proposed quadrex algorithm examines the behaviour of the quadratic objective function near the origin or a translate of the origin. If that origin is feasible, then it is the optimal solution. Otherwise, the point of intersection of the functions in the constraint set closest to that origin is the optimal solution. The method works whenever the matrix Q of the quadratic form in the objective function is strictly positive-definite with no negative eigenvalue.

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